# EXAMPLES IN MATHEMATICS

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**Abstract**

This is a notes of examples. I write it mainly because I think I am not good at remembering and using examples in mathematics.

# Contents

# 1 Analysis

## 1.1 Functional Analysis

$f \in C^{\infty}, \forall x \in \mathbb{R}, \exists n = n(x) \in \mathbb{N}^*, s.t. f^{(n)}(x) = 0.$
(From MSE:If $f$ is infinitely differentiable then f coincides with a polynomial)

*Proof.* $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$, hence it suffices to show $f$ is a polynomial on $[-n, n]$.

The proof is by contradiction. Suppose $f$ is not a polynomial on $[-n, n]$.

Let: $S_n = \{x : f^{(n)}(x) = 0\}, n \in \mathbb{N}^*$, then $S_n$ is closed since $f^{(n)}(x)$ is continuous.

Let $X = \{x : \forall (a, b)$ containing $x, f|_{(a,b)}$ is not a polynomial.$\}$. $X$ is a non-empty closed set without isolated points:

X is closed: $X^c = \{x; \exists (a, b)$ containing $x, f|_{(a,b)}$ is a polynomial$\}$, hence $X^c$ is clearly open. $X$ is non-empty: it suffices to show $X^c \neq [-n, n]$, otherwise we can find an open cover of $[-n, n]$, hence a finite subcover of $[-n, n]$ such that f is a polynomial on them respectively, then there exists a sufficiently large N, such that $f'(x) = 0$, hence $f$ is a polynomial on $[-n, n]$, which is a contradiction. X is without isolated points: otherwise $\exists a(\text{if } \neq n, -n) \in X$, such that $\exists \epsilon > 0$, $(a - \epsilon, a) \cup (a, a + \epsilon) \subset X^c$,$(a - \epsilon, a) \cup (a, a + \epsilon)$ is bounded, so on every compact subset, $f$ is a polynomial, hence $f$ is a polynomial on $(a - \epsilon, a)$,$(a, a + \epsilon)$ respectively. Since $f \in C^{\infty}$, it is easy to show $\exists N$ sufficiently large such that $f^{(N)}(x) = 0, \forall x \in (a - \epsilon, a + \epsilon)$. Then $a \in X^c$, which is a contradiction. For $a = n, -n$, we can easily get a contradiction by the same argument.

$X = \bigcup_{n=1}^{\infty} (S_n \cap X)$, by Baire Category theorem, $\exists k \in \mathbb{N}^*$, such that $S_k \cap X$ is not a nowhere dense set, i.e. $(S_k \cap X)^{\circ}$ is not empty (in the induced topology of $X$). Hence $\exists$ interval $(a, b)$, such that $(a, b) \cap X \subset S_k \cap X$ and $(a, b) \cap X$ is non-empty. Since $X$ is without isolated point, every point of $(a, b) \cap X$ is an accumulation point. Hence $(a, b) \cap X \subset S_n, \forall n \geq k$(we can easily prove by the definition of derivative and $S_n$).

Now consider a maximal interval $(c, e) \subset (a, b) \backslash X$, then we can easily show $f$ is a polynomial, of degree $d$. Then $f^{(d)} = \text{const.}$ ,$d < k$ since $c$ or $e$ is in X hence in $S_k$. So $f^{(k)} = 0$, which is a contradiction to $(a, b) \cap X$ is non-empty. □

## 1.2 Measure

### 1.2.1 A partition of Interval

$T$ is measurable, $\mu$ is a finite measure, then $\forall E$ measurable

$$\lim_{n \to} \mu(T^{-n} E \backslash \bigcup_{k=0}^{n-1} T^{-k} E) = 0$$

Hint:note that, we can define

$$E_k = \{x : x \in E, x \notin T^{-1}E, ..., x \notin T^{-(k-1)}E, x \in T^{-k}E\}$$

$$E_k^* = \{x : x \notin E, x \notin T^{-1}E, ..., x \notin T^{-(k-1)}E, x \in T^{-k}E\}$$

#### 1.2.2 Dirac measure

$(X, 2^X)$ is a measure space, $\delta_x$ is a measure:

$$\delta_x(A) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases} \tag{1}$$

$$m = \sum_{n=1}^{N} \delta_{T^i x} \text{ is ergodic, where T is an endomorphism, and } T^N x = x. \tag{2}$$

#### 1.2.3 An outer measure which is not continuous from below

$(\mathbb{Z}, 2^{\mathbb{Z}}, \mu^*)$,where:

$$\mu^*(A) = \begin{cases} 1, & A \text{ is finite and nonempty} \\ 0, & A = \emptyset \\ \infty, & A \text{ is infinite} \end{cases} \tag{3}$$

### 1.3 Ergodic Theory

#### 1.3.1 Bernoulli shift is strong mixing

$Y = \{0, 1, ..., d\}, X = Y^{\mathbb{N}_0}$, $\mathcal{A}$ is the algebra generated by measurable rectangles, ${}_l[a]_k = \{x \in X : x_i = a_i, l \leq i \leq k\}$,then for all $A, B \in \sigma(\mathcal{A})$, T is the shift,then T is strong mixing, i.e. $\lim_{n\to\infty} m(T^{-n}A \cap B) = m(A)m(B)$.

#### 1.3.2 Rotation of torus

T is a measure-preserving transformation(Hint: construct an algebra which generates the Borel algebra), $m = Leb|_{[0,1)}$.

$$\begin{aligned} T :&[0,1) \to [0,1) \\ &x \longmapsto x + \alpha \end{aligned} \tag{4}$$

**Proposition 1.1.** T is ergodic if and only if $\alpha$ is irrational.

*Proof.* "$\Longrightarrow$": We prove this by contradiction, i.e. suppose $\alpha = \frac{q}{p}, p, q \in \mathbb{N}$, define

$$\begin{aligned} f : [0,1) &\longmapsto [0,1) \\ x &\longmapsto px \end{aligned} \tag{5}$$

therefore, $f \circ T = f$, by Walters(GTM 79) theorem 1.6, $f =$ const., which is a contradiction.

"$\Longleftarrow$": We prove: $\forall f \in L^2(m), f \circ T = f$ implies $f =$ const. Then by Walters(GTM 79) theorem 1.6, $f$ is ergodic. It is easy to prove this by using the Fourier expansion of f.

Remark: $f \in L^p$, then the Fourier series of $f$ converges almost everywhere if $1 < p < \infty$; but it is not true for $p = 1$. □

### 1.3.3 Koopman operator

$(X, \mathcal{B}, m, T)$ is a measure preserving system, the koopman operator:

$$\begin{aligned} U_T : L^0(m) &\longrightarrow \quad L^0(m) \\ f &\longmapsto \quad f \circ T \end{aligned} \tag{6}$$

It has the following property:

$$\int f \mathrm{d}m = \int f \circ T \mathrm{d}m$$

### 1.3.4 Shift space is Lipschitz homeomorphic to a cantor set

### 1.3.5 Adding Machine

$\Sigma^d$ is minimal under $T$, where T is an addition operator which can be seen as "+1" with number written in a inverse order. I write this because it appears in "Conformal fractals" and there is a mistake in the book.

### 1.3.6 Complex linear fractals(Cantor set)

This is a generalized form of cantor set.(It is from "Conformal fractals", it makes something simple look much more difficult and strange for beginners like me!)

Let $U \subset \mathbb{C}$ be an open connected set.$T_i(z) = \lambda_i z + a_i, \lambda_i \in \mathbb{C}, |\lambda| < 1, a_i \in \mathbb{C}, i \in \{1, ..., n\}$, such that $\overline{T_i(U)}$ are pairwise disjoint and contained in $U$.

Define the limit cantor set:

$$\Lambda = \bigcap_{k \geq 0} \bigcup_{(i_0, \ldots, i_k)} T_{i_0} \circ ... \circ T_{i_k}(U) = \bigcup_{(i_0, \ldots, i_k, \ldots)} \lim_{k \to \infty} T_{i_0} \circ ... \circ T_{i_k}(z)$$

Note that

$$\lim_{k \to \infty} T_{i_0} \circ ... \circ T_{i_k}(z) = \sum_{k=1}^{\infty} \lambda_{i_0} ... \lambda_{i_{k-1}} a_k$$

So this equality is easy to verify and the definition is independent of x.

## 1.4 Ordinary Differential Equation

### 1.4.1 An autonomous system with exactly one limit cycle

This is from Arnold's Ordinary Differential Equation P73.

$$\begin{cases} \dot{x} = y + x(1 - x^2 - y^2) \\ \dot{y} = -x + y(1 - x^2 - y^2) \end{cases} \tag{7}$$

By passing to polar coordinates, we obtain:

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = -1 \end{cases} \tag{8}$$

The phase curve is as follows:

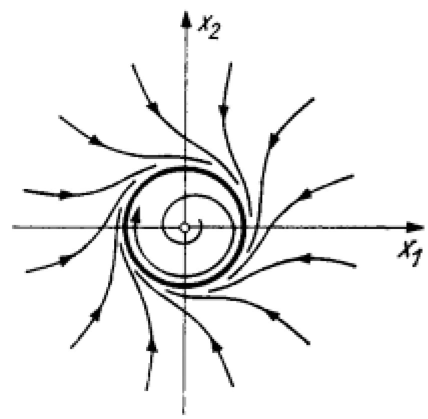


Figure 1: Integral curves in the (x,y)-plane

# 2 Algebra

## 2.1 Abstract Algebra

### 2.1.1 The field of Laurent series and the rational function field

The elements of field of Laurent series $k((x))$ (where $k$ is the field) is the formal infinite sums $\sum\limits_{n=n_0}^{+\infty} a_n x^n$, where $a_n \in k$ and $n_0 \in \mathbb{Z}$. The rational function field field can be viewed as a subfield of the field of Laurent series.

This is because any element in the rational function field $k(x)$ is of the form $\frac{f(x)}{g(x)}$, where $f(x)$, $g(x) \in k[x]$, then $\frac{f(x)}{g(x)} = f(x)[g(x)]^{-1}$.

## 2.2 Homological Algebra

### 2.2.1 R-module R is projective but not injective

R-module R is projective, because $\mathrm{Hom}_R(R, N) \cong R$.

R-module R is not injective, consider an exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{projection} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Hom($\mathbb{Z}$, .) is a left exact functor, we have the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{f} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$$

It suffices to show $f$ is note surjective: for $id \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$, $id \notin f(\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}))$, otherwise $id(1) = f \circ g(1) \neq 1, g \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$.

### 2.2.2 R-module R is injective if R is a field

Note that when R is a field, we can consider this proposition in the category of vector space.

### 2.3 Commutative Algebra

#### 2.3.1 A ring which is not Noetherian but has a Noetherian prime spectra

$R = k[x_1, x_2, ..., x_n, ...]$ is a polynomial ring with infinite indeterminates, and $I = (x_1, x_2^2, ..., x_n^n, ...)$ is an ideal. Then $S = R/I$ is a ring which is not Noetherian but has a Noetherian prime spectra.

$J = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n, ...)$ is and ideal of $S$. $S/J \cong k$, hence $J$ is a maximal ideal; every element of $J$ is nilpotent, hence $J$ is contained in the nilradical, which is the intersection of all prime ideals of $J$. Therefore, $J$ is the unique prime ideal of $S$, hence the prime spectra of $S$ is Noetherian.There exists an strictly increasing sequence of ideals of $S$:$(\bar{x}_1) \subset (\bar{x}_1, \bar{x}_2) \subset ... \subset (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \subset ...$

## 3 Geometry

### 3.1 Point Set Topology

#### 3.1.1 Projections from product space is not necessarily closed

$p : \mathbb{R}^2 \to \mathbb{R}$ is not closed. For $p\{(x, y) : xy = 1\} = R\backslash\{0\}$.

#### 3.1.2 A continuous map which is bijectvie but not homeomorphic

$f : \mathbb{E}^1\backslash[0, 1) \to \mathbb{E}^1$

$$f(x) = \begin{cases} x, & x < 0 \\ x - 1, & x \geq 1 \end{cases} \tag{9}$$

#### 3.1.3 A continuous map which is closed but not open

$$\begin{aligned} f : \mathbb{E}^1 &\longmapsto \mathbb{E}^1 \\ x &\longmapsto 1 \end{aligned} \tag{10}$$

#### 3.1.4 A continuous map which is open but not closed

Let the open set of $\mathbb{R}$ be $\emptyset, \{1\}, \mathbb{R}$.

$$\begin{aligned} f : \mathbb{E}^1 &\longmapsto \mathbb{R} \\ x &\longmapsto 1 \end{aligned} \tag{11}$$

#### 3.1.5 An example which is $T_2$ but not $T_3$, is $C_1$ and separable but not $C_2$

This example is in You's Basic Topology P44 Ex.18.

Let $S = \mathbb{R}\backslash\mathbb{Q}$,the topology of $\searrow$ is $\tau = \{U\backslash A|\text{U is open in } \mathbb{E}^1, A \subset S\}$.

**(1)**. $\tau$ is a topology:

Clearly $\emptyset, \mathbb{R} \in \tau$.

Finite intersection:

$$U_n \in \mathbb{E}^1, A_n \subset S, n \in \{1, 2, ..., N\}, \bigcap_{n=1}^{N} U_n\backslash A_n = (\bigcap_{n=1}^{N} U_n)\backslash(\bigcup_{n=1}^{N} A_n) \in \tau$$

Arbitrary union:

$$U_i \in \mathbb{E}^1, A_i \subset S, i \in \Lambda, \bigcup_{i\in\Lambda}(U_i \backslash A_i) \subset (\bigcup_{i\in\Lambda} U_i)\backslash(\bigcap_{i\in\Lambda} A_i) \Longrightarrow \bigcup_{i\in\Lambda}(U_i\backslash A_i) \in \tau$$

**(2)**.$(\mathbb{R}, \tau)$ is $T_2$ but not $T_3$.

This is because the open set in $\mathbb{E}^1$ is also open in $\tau$, so it is $T_2$. For $(a,b)\backslash S$ and $r \in (a,b) \cap S$, we cannot find two open set separating $(a,b)$ and $\{r\}$, hence it is not $T_3$.

**(3)**.$(\mathbb{R}, \tau)$ is $C_1$ and separable.

For the neighbourhood basis of $r \in \mathbb{R}$ is $\{(r - q_n, r + q_n)\backslash T\}_{q\in Q}$, For $r \notin \mathbb{Q}, T = S$ if $r \in \mathbb{Q}$, $T = S\backslash\{r\}$, if $r \in S$.

$(\mathbb{R}, \tau)$ is separable, i.e. has a countable dense subset, because $\overline{\mathbb{Q}} = \mathbb{R}$.

**(4)**.The induced topology of $S(\tau_S)$ from $\tau$ is discrete.

$\forall p \in S, p \in [\mathbb{R}\backslash(S\backslash\{p\})] \cap S = \{p\}$.

**(5)**.$(\mathbb{R}, \tau)$ is not $C_2$.

Otherwise $S, \tau_S$ would be $C_2$

### 3.1.6 Topologist's sine curve

$A = \{(x, \sin\frac{1}{x}); x \in (0,1]\}, \bar{A} = \overline{\{(x, \sin\frac{1}{x}); x \in (0,1]\}}$ is connected but neither path-connected nor locally connected.

$\bar{A}$ is connected: $A \cong (0,1]$ is a connected and dense subset of $\bar{A}$.

$\bar{A}$ is not locally connected: $(0,0) \in U = \bar{A}\backslash\{(x,y)|y = 1\} \subset \bar{A}$ contains no connected neighbourhood.

$\bar{A}$ is not path-connected: it suffices to show $\partial A$ is a path-connected component. Clearly $\partial A$ is path connected, we can prove $\partial A$ is a path-connected component of A by the above argument. Suppose $a$ is a path such that $a(0) \in A$, $J = a^{-1}(A)$. Then $J$ is closed and nonempty. It suffices to show J is open, then $J = I$, hence $a(I) \subset A$, then $A$ is a path connected component. $\forall t \in J, a(t) \in A$, without loss of generosity, suppose $a(t) \neq (0,1)$, then $a(t) \in U$. There exist path-connected neighbourhood of $t$, such that $a(W) \subset U$, since $[0,1]$ is locally path connected. Since the continuous image of a path-connected set is path-connected, $a(W) \subset A$,therefore $W \subset a^{-1}(A)$. Hence $a^{-1}(A)$ is both open and closed. Hence $A$ is a path connected component. Therefore $\bar{A}$ is not path-connected.

Remark: $\{(x, \sin\frac{1}{x}) : x \neq 0\}$ is a smooth manifold.

## 3.2 Algebraic Topology

# 4 Applied Mathematics

## 4.1 Asymptotic Methods and Perturbation Theory

The last problem of 1.5 from Introduction to Perturbation Methods: suppose $f = o(\phi)$, for small $\epsilon$, where $f$ and $\phi$ are continuous functions. Give an example to show that it is not necessarily true that

$$\int_0^\epsilon f = o(\int_0^\epsilon g)$$

Example:

$$f(x) = x^2|\sin(\frac{1}{x})|$$

$$\phi(x) = x\sin(\frac{1}{x})$$

$f = o(\phi)$, but $\int_0^\epsilon f \neq o(\int_0^\epsilon \phi)$:

$\int_0^\epsilon f > 0, \forall \epsilon > 0$, but $\int_0^\epsilon g$ has infinitely many zeros in any neighbourhood of 0.

Let $F(x) = \int_0^x x\sin(\frac{1}{x})dx$,

$$F(\frac{1}{n\pi}) = \int_0^{\frac{1}{n\pi}} x\sin(\frac{1}{x})dx = \frac{(-1)^n}{(n\pi)^3} - 3\int_0^{\frac{1}{n\pi}} x^2\cos(\frac{1}{x})dx$$

Note that $|3\int_0^{\frac{1}{n\pi}} x^2\cos(\frac{1}{x})dx| \leq 3\int_0^{\frac{1}{n\pi}} |x^2\cos(\frac{1}{x})|dx < 3\int_0^{\frac{1}{n\pi}} x^2dx = \frac{1}{(n\pi)^3}$

So $F(\frac{1}{n\pi})(-1)^n > 0$, so $o(\int_0^\epsilon g)$ has infinitely many zeros in any neighbourhood of 0, so $\int_0^\epsilon f \neq o(\int_0^\epsilon \phi)$.